Parisi solutions for the anisotropic spin glass problem

1982 J. Phys. A: Math. Gen. 15 L437

(http://iopscience.iop.org/0305-4470/15/8/011)

J. Phys. A: Math. Gen. 15 (1982) L437–L442. Printed in Great Britain

LETTER TO THE EDITOR

Parisi solutions for the anisotropic spin glass problem

David Elderfield and David Sherrington
Blackett Laboratory, Imperial College, London SW7 2BZ, UK

Received 17 May 1982

Abstract. A Heisenberg spin glass with local uniaxial anisotropy is studied at mean field level in terms of a Parisi symmetry breaking scheme. It is shown that replica symmetry breaking leads to important modifications to the replica symmetric expressions for the susceptibilities χ_T, χ_L which signal the transitions in this system.

Recently it has been demonstrated (Cragg and Sherrington 1982, Roberts and Bray 1982) that a vector spin glass with single ion uniaxial anisotropy provides an interesting phase diagram and is readily accessible experimentally. In particular the model is applicable to spin glass systems in which the host has an HCP structure; for example, *Zn* Mn, *Cd* Mn (Albrecht *et al* 1982). Discussion of the low-temperature spin glass phases is, however, complicated by the need to break replica symmetry (de Almeida and Thouless 1978, Cragg *et al* 1982). Whilst Parisi (1979a, b) has presented a generally accepted description for the Ising model, little progress has yet been made for vector models. Here we describe a Parisi-like scheme for the O(m) symmetric model in the presence of uniaxial anisotropy. Solutions for the mean field equations are presented, and some consequences for the longitudinal and transverse susceptibilities are discussed.

To facilitate the construction of a mean field description we follow Sherrington and Kirkpatrick (1975) and adopt a Hamiltonian

$$H = -\sum_{i,j} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - \sum_i D S_{i1}^2 \tag{1}$$

which describes the interaction of m-dimensional classical vectors ($|\boldsymbol{S}|^2 = m$) coupled via a set of infinite-range exchanges J_{ij}, each of which is an independent random variable with probability distribution

$$P(J_{ij}) = (N/2\pi J^2)^{1/2} \exp(-NJ_{ij}^2/2J^2). \tag{2}$$

The scaling of the variance as N^{-1} ensures, as usual, a sensible thermodynamic limit. The final term in (1) represents a uniaxial anisotropy with the '1' direction an easy (hard) axis for $D > 0$ ($D < 0$).

Using the replica trick we obtain the free energy per spin in the thermodynamic limit in the form

$$f = -T \lim_{n\to 0} n^{-1} \max\{F(q^{\alpha\beta}, p^{\alpha\beta}, x^{\alpha})\} \tag{3}$$

where the free energy functional to be maximised is given by

$$F(q^{\alpha\beta}, p^{\alpha\beta}, x^{\alpha})$$

$$= -\tfrac{1}{2}\beta^2 \Big(\sum_{(\alpha\beta)} (m-1)(q^{\alpha\beta})^2 + (p^{\alpha\beta})^2 \Big) - \tfrac{1}{2}\beta^2 \Big(\sum_{\alpha} m(m-1)(x^{\alpha})^2 + mx^{\alpha} \Big)$$

$$- \ln \Big\{ \operatorname*{Tr}_{\{S^{\alpha}\}} \exp \Big[\beta^2 \sum_{(\alpha\beta)} \Big(q^{\alpha\beta} \sum_{\lambda \neq 1}^{m} S_{\lambda}^{\alpha} S_{\lambda}^{\beta} + p^{\alpha\beta} S_1^{\alpha} S_1^{\beta} \Big)$$

$$+ \frac{\beta}{2} \sum_{\alpha} (2D + \beta m x^{\alpha})(S_1^{\alpha})^2 \Big] \Big\}. \tag{4}$$

Here the superscripts $\alpha, \beta = 1, \ldots, n$ are replica labels; the notation (α, β) relating to a sum over distinct pairs. For simplicity units have been chosen such that $J = k_B = 1$.

The replica symmetric (RS) solution has

$$x^{\alpha} = x \equiv (\overline{\langle S_1^2 \rangle} - 1)/(m-1),$$

$$q^{\alpha\beta} = q \equiv \overline{\langle S_{\lambda} \rangle^2} = 1 - x - T\chi_T, \qquad \text{all } \alpha, \beta; \alpha \neq \beta, \tag{5}$$

$$p^{\alpha\beta} = p \equiv \overline{\langle S_1 \rangle^2} = 1 + (m-1)x - T\chi_L$$

where $\langle\ \rangle$, $\overline{}$ denote thermal and disorder averages and χ_T, χ_L are respectively the transverse and longitudinal susceptibilities in the infinitesimal field limit. The RS phase diagram has already been discussed (Cragg and Sherrington 1982, Roberts and Bray 1982); briefly, one may identify transverse T ($q \neq 0$, $p = 0$), longitudinal L ($q = 0, p \neq 0$) and mixed LT ($q, p \neq 0$) spin glass phases, whilst the uninteresting quadrupolar order parameter x is non-zero throughout the cut DT plane ($D \neq 0$), see figure 1.

Unfortunately the RS solution is thermodynamically unstable (de Almeida and Thouless 1978, Cragg and Sherrington 1982, Roberts and Bray 1982), so predictions based on (5) for the low-temperature phases (T, L, LT) must be treated with caution; although clearly the phase diagram is qualitatively correct. The discovery of this instability for the Ising model has led a number of authors (Bray and Moore 1978, Blandin *et al* 1980, Parisi 1979a, b) to construct solutions with broken RS but, for various reasons, only that of Parisi is generally accepted. We have therefore developed a Parisi-like scheme for the O(m), $D \neq 0$ model which, near the critical point $D \neq 0$, $T = 1$ takes the form

$$f = -T \max\{F(\{q(r)\}, \{p(r)\}, x)\} \tag{6}$$

where the free energy functional F is given by

$$F(\{q(r)\}, \{p(r)\}, x)$$

$$= \tfrac{1}{4}(m-1)\Big[\tau + \frac{2my}{(m+2)} + \Big(\frac{m}{m+2}\Big)^2 (2m^2 - m - 12)y^2\Big] \int_0^1 dr\, q^2(r)$$

$$+ \tfrac{1}{4}\Big[\tau - 2\frac{m(m-1)y}{(m+2)} - 3\Big(\frac{m}{m+2}\Big)^2 (m-1)\frac{(m^2+m-4)y^2}{(m+4)}\Big] \int_0^1 dr\, p^2(r)$$

$$- \tfrac{1}{4}m(m-1)y^2\Big[\tau + \frac{2}{(m+2)}\Big] + (m-1)\frac{Dy}{\beta} + \frac{1}{6}\frac{m^3(m-1)(m-2)y^3}{(m+2)(m+4)(m+6)}$$

$$+ \tfrac{1}{6}\Big(1 + 3\frac{m(m-1)y}{(m+2)}\Big)\Big[\int_0^1 dr \Big(rp^3(r) + 3p(r)\int_0^r dw\, p^2(w)\Big)\Big]$$

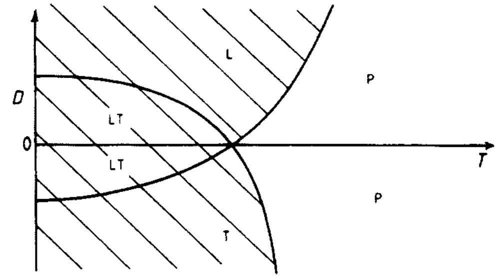


Figure 1. Schematic phase diagram for a vector spin glass with uniaxial anisotropy. The hatched area represents that part of the phase diagram for which replica symmetry is broken.

$$
\begin{aligned}
&+\tfrac{1}{6}(m-1)\left(1-\frac{3my}{(m+2)}\right)\left[\int_0^1 \mathrm{d}r\left(rq^3(r)+3q(r)\int_0^r \mathrm{d}w\, q^2(w)\right)\right]\\
&-\tfrac{1}{8}\left[\int_0^1 \mathrm{d}r\left((r^2+1)p^4(r)+4p^2(r)\int_0^r \mathrm{d}w\, p^2(w)+4p(r)\int_0^r \mathrm{d}w\, wp^2(w)\right.\right.\\
&\left.\left.+12p(r)\int_0^r \mathrm{d}w\, p(w)\int_0^w \mathrm{d}z\, p^2(z)\right)\right]\\
&-\tfrac{1}{8}(m-1)\left[\int_0^1 \mathrm{d}r\left((r^2+1)q^4(r)+4q^2(r)\int_0^r \mathrm{d}w\, q^2(w)\right.\right.\\
&\left.\left.+4q(r)\int_0^r \mathrm{d}w\, wq^2(w)+12q(r)\int_0^r \mathrm{d}w\, q(w)\int_0^w \mathrm{d}z\, q^2(z)\right)\right]\\
&-\tfrac{1}{8}(m^2-2m-2)\int_0^1 \mathrm{d}r\, p^4(r)+\tfrac{1}{8}(m-1)(m^2-2)\int_0^1 \mathrm{d}r\, q^4(r)\\
&+\frac{1}{4}\left(\frac{m-1}{m+2}\right)\left[\left(\int_0^1 \mathrm{d}r\, p^2(r)\right)^2+\int_0^1 \mathrm{d}r\, p^4(r)\right]\\
&+\frac{1}{4}\left(\frac{m-1}{m+2}\right)\left[\left(\int_0^1 \mathrm{d}r\, q^2(r)\right)^2+\int_0^1 \mathrm{d}r\, q^4(r)\right]+\frac{1}{2}\frac{m^2-1}{(m+2)^2}\int_0^1 \mathrm{d}r\, p^2(r)q^2(r)\\
&-\frac{1}{2}\left(\frac{m-1}{m+2}\right)\left[\left(\int_0^1 \mathrm{d}r\, p^2(r)\right)\left(\int_0^1 \mathrm{d}r\, q^2(r)\right)\right.\\
&\left.+\int_0^1 \mathrm{d}r\, q^2(r)p^2(r)\right]+\mathrm{O}(p^5, q^5, \ldots)
\end{aligned}
\tag{7}
$$

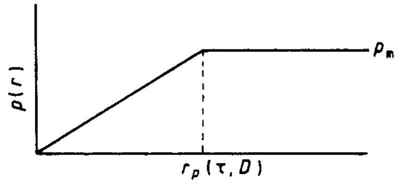
and the order parameters $q(r)$, $p(r)$ are now real functions on the interval (0, 1). We have absorbed a factor T^{-2} into q, p, x and $\tau = T^2-1$, $y = x + 2D/mT$.

Solving (6) and (7), the qualitative structure of the phase diagram near the isotropic critical point is confirmed. Predictably, the P–T, P–L phase boundaries are given correctly by RS theory. Perhaps surprisingly, we also find that RS breaking does not move the embedded boundaries T–LT, L–LT to $\mathrm{O}(\tau^2, D^2)$.

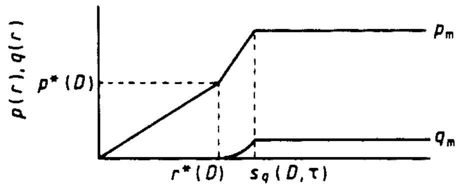
Let us now discuss explicitly the solutions for $q(r)$, $p(r)$ for $r \in (0, 1)$. Consider first $D > 0$. As τ is lowered progressively from high temperatures we find

(i) $\tau > \tau_p(D)$; the phase is paramagnetic; $q = p = 0$;

(ii) $\tau_p(D) > \tau > \tau_q(D)$; the system exhibits longitudinal spin glass ordering; $q = 0$ whilst $p(r)$ is of the form

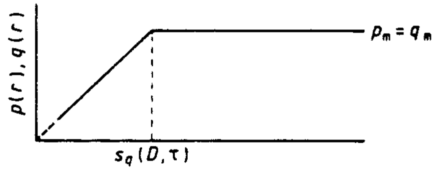


(iii) $\tau_q(D) \gtrsim \tau$; the system has entered the mixed phase; $q, p \neq 0$.



where as τ approaches $\tau_q(D)$ from below $p_m \to p^*$, $q_m \to 0$, $s_q \to r^*$. Observe that p^*, r_p^* are independent of τ and thus effectively constant.

(iv) $\tau_q(D) \gg \tau$; finally the system crosses over to the isotropic solution; $p(r) \cong q(r)$.



where effects due to the anisotropy are confined to a small domain O(D) at the origin $r = 0$. For $D < 0$ the roles of p, q are simply reversed, so we shall not discuss this case in detail; instead we tabulate the various characteristic parameters $p^*, q^*, \ldots$ in table 1.

The thermodynamic susceptibilities are calculated by applying a small field $\boldsymbol{h}$ and using

$$\chi_{\mu\nu} = -\partial^2 f(\boldsymbol{h})/\partial h_\mu \partial h_\nu \tag{8}$$

where $f(\boldsymbol{h})$ is the analogue of (6) in the field. In the limit $\boldsymbol{h} \to 0$ this yields (cf Parisi 1980a, b, Thouless *et al* 1980)

$$T\chi_{\rm L} = 1 + T^2\left((m-1)x - \int_0^1 \mathrm{d}r\, p(r)\right), \qquad T\chi_{\rm T} = 1 - T^2\left(x + \int_0^1 \mathrm{d}r\, q(r)\right). \tag{9}$$

By contrast,

$$\overline{\langle S_1\rangle^2} = T^2 \max p(r) \equiv T^2 p_{\rm m}, \qquad \overline{\langle S_\lambda\rangle^2} = T^2 \max q(r) \equiv T^2 q_{\rm m}, \tag{10}$$

so that the standard linear response formula

$$T\chi_{\mu\nu} = \overline{\langle S_\mu S_\nu\rangle} - \overline{\langle S_\mu\rangle\langle S_\nu\rangle} \tag{11}$$

gives predictions different from (9), indicating a breakdown of the ergodic theorem. The difference between the two forms of susceptibility may be interpreted as a measure of the irreversibility.

Table 1.

$D>0$	
$\tau_q<\tau<\tau_p$	$p_{\mathrm{m}}=-\tau/2+(m-1)DT+\mathrm{O}(D^2,\tau^2)\equiv-\frac{1}{2}(\tau-\tau_p)+\dots$
	$r_p=-\dfrac{9}{(m+2)^2}(\tau-\tau_p)$
$\tau\lesssim\tau_q$	$p_{\mathrm{m}}=-\frac{1}{2}(\tau-\tau_p)+\dots,\qquad q_{\mathrm{m}}=-\tau/2+DT+\mathrm{O}(D^2,\tau^2)=-\frac{1}{2}(\tau-\tau_q)+\dots$
	$p^*=mDT,\qquad r^*=\dfrac{18mDT}{(m+2)^2}$
	$s_q=r^*+\dfrac{18}{(m+2)^2}(p_{\mathrm{m}}-p^*)-\dfrac{10}{9}\dfrac{q_{\mathrm{m}}}{(p_{\mathrm{m}}-p^*)}\left(\dfrac{m-1}{m+2}\right)^2 \qquad \tau\simeq\tau_q$
$\tau\ll\tau_q$	$p_{\mathrm{m}}\cong q_{\mathrm{m}}=-\tau/2,\qquad s_q=-3\tau/(m+2)$
$D<0$	
$\tau_p<\tau<\tau_q$	$q_{\mathrm{m}}=-\tau/2+DT+\mathrm{O}(D^2,\tau^2)\equiv-\frac{1}{2}(\tau-\tau_q)+\dots$
	$r_q=-\dfrac{6(m+1)}{(m+2)^2}(\tau-\tau_q)$
$\tau\lesssim\tau_p$	$q_{\mathrm{m}}=-\frac{1}{2}(\tau-\tau_q)+\dots$
	$q^*=\left(\dfrac{m}{m-1}\right)DT,\qquad r^*=\dfrac{6m(m+1)}{(m-1)(m+2)^2}DT$
	$s_p=r^*+\dfrac{6(m+1)}{(m+2)^2}(q_{\mathrm{m}}-q^*)-\dfrac{10p_{\mathrm{m}}}{3(m+1)(q_{\mathrm{m}}-q^*)}\left(\dfrac{m-1}{m+2}\right)^2 \qquad \tau\simeq\tau_p$
$\tau\ll\tau_p$	$p_{\mathrm{m}}\cong q_{\mathrm{m}}=-\tau/2,\qquad s_p=-3\tau/(m+2)$

For $D>0$, near the isotropic critical point the (Parisi) thermodynamic susceptibilities may be developed in the form

$$T\chi_{\mathrm{L}}=1+(m-1)T^2x+\begin{cases}0 & \tau>\tau_p,\\ \dfrac{T^2}{2}\dfrac{(\tau-\tau_p)}{[1-(m-1)^2DT]}\left[1-\dfrac{1}{4}\dfrac{(3m+12)}{(m+2)}(\tau-\tau_p)\right]+\mathrm{O}(D^3,\tau^3) & \tau<\tau_p,\end{cases}$$

$$T\chi_{\mathrm{T}}=1-T^2x+\begin{cases}0 & \tau>\tau_q,\\ \dfrac{T^2}{2}\dfrac{(\tau-\tau_q)}{[1+(m-1)DT]}\left[1-\dfrac{1}{4}\left(\dfrac{5m+8}{m+2}\right)(\tau-\tau_q)\right] & \\ \quad-\dfrac{T^2}{2(m+2)}(\tau-\tau_q)[\frac{1}{2}(\tau-\tau_p)-mDT]+\mathrm{O}(D^3,\tau^3) & \tau<\tau_q,\end{cases}$$

so that χ_{T}, χ_{L} respectively exhibit characteristic cusps at the q, p ordering temperatures. Throughout the domain of interest the quadrupolar order parameter x is non-zero and given to this order by

$$y \equiv x + \frac{2D}{mT}$$

$$= \left(\frac{m+2}{m}\right) DT \left(1 + \frac{2m(m-2)DT}{(m+2)(m+4)(m+6)}\right) \Big/ \left(1 + \tau \frac{(m+2)}{2}\right) + \frac{1}{(m+2)} (q_m^2 - p_m^2).$$

As one might expect, all the susceptibilities obey an inequality of the form

$$\chi \geqslant \chi(\text{RS}).$$

However, only for the Ising limit do we observe the extreme flatness seen by Parisi,

$$\chi\ (\text{Ising, Parisi}) = \chi(T_c) + O(t^3),$$

where $t = 1 - T/T_c$ is the reduced temperature.

For completeness, we note finally the results for $D < 0$ in the RS broken scheme;

$$T\chi_L = 1 + (m-1)T^2 x + \begin{cases} 0 & \tau > \tau_p, \\ \dfrac{T^2}{2} \dfrac{(\tau - \tau_p)}{[1 - (m-1)^2 DT]} \left[1 - \dfrac{1}{4}\left(\dfrac{3m+12}{m+2}\right)(\tau - \tau_p)\right] & \\ -\dfrac{T^2}{2} \dfrac{(m-1)}{(m+2)} (\tau - \tau_p) \left[\tfrac{1}{2}(\tau - \tau_q) - \left(\dfrac{m}{m-1}\right)^2 DT\right] & \tau < \tau_p, \end{cases}$$

$$T\chi_T = 1 - T^2 x + \begin{cases} 0 & \tau > \tau_q, \\ \dfrac{T^2}{2} \dfrac{(\tau - \tau_q)}{[1 + (m-1)DT]} \left[1 - \dfrac{1}{4}\left(\dfrac{5m+8}{m+2}\right)(\tau - \tau_q)\right] + O(\tau^3, D^3) & \tau < \tau_q. \end{cases}$$

Financial support from the SERC is gratefully acknowledged.

References

Albrecht H, Wassermann E, Hedgecock F T and Monod P 1982 *Phys. Rev. Lett.* **48** 819–22
de Almeida J R L and Thouless D J 1978 *J. Phys. A: Math. Gen.* **11** 983–90
Blandin A, Gabay M and Garel T 1980 *J. Phys. C: Solid State Phys.* **13** 403–18
Bray A J and Moore M A 1978 *Phys. Rev. Lett.* **41** 1068–72
Cragg D M and Sherrington D 1982 to be published
Cragg D M, Sherrington D and Gabay M 1982 to be published
Parisi G 1979a *Phys. Lett.* A **73** 203–5
—— 1979b *Phys. Rev. Lett.* **23** 1754–6
—— 1980a *J. Phys. A: Math. Gen.* **13** 1101–12
—— 1980b *Phil. Mag.* **41** 667–80
Roberts S A and Bray A J *J. Phys. C: Solid State Phys.* **15** L527–31
Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **32** 1792
Thouless D, de Almeida J R L and Kosterlitz J M 1980 *J. Phys. C: Solid State Phys.* **13** 3271–80